Chapter 1. Combinatorial Theory

1.2: More Counting

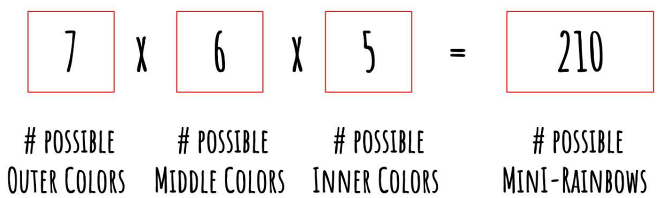
Slides (Google Drive) Video (YouTube)

1.2.1 k-Permutations

Last time, we learned the foundational techniques for counting (the sum and product rule), and the factorial notation which arises frequently. Now, we'll learn even more "shortcuts"/"notations" for common counting situations, and tackle more complex problems.

We'll start with a simpler situation than most of the exercises from last time. How many 3-color mini rainbows can be made out of 7 available colors, with all 3 being different colors?

We choose an outer color, then a middle color and then an inner color. There are 7 possibilities for the outer layer, 6 for the middle and 5 for the inner (since we cannot have duplicates). Since order matters, we find that the total number of possibilities is 210, from the following calculation:



Let's manipulate our equation a little and see what happens.

$$
\begin{aligned}
7 \cdot 6 \cdot 5 &= \frac{7 \cdot 6 \cdot 5}{1} \cdot \frac{4 \cdot 3 \cdot 2 \cdot 1}{4 \cdot 3 \cdot 2 \cdot 1} && \text{[multiply numerator and denominator by } 4! = 4 \cdot 3 \cdot 2 \cdot 1\text{]} \\
&= \frac{7!}{4!} && \text{[def of factorial]} \\
&= \frac{7!}{(7-3)!}
\end{aligned}
$$

Notice that we are "picking" 3 out of 7 available colors - so order matters. This may not seem useful, but imagine if there were 835 colors and we wanted a rainbow with 135 different colors. You would have to multiply 135 numbers, rather than just three!

Definition 1.2.1: k-Permutations

If we want to arrange **only** k out of n distinct objects, the number of ways to do so is $P(n, k)$ (read as "n pick k"), where

$$P(n,k) = n \cdot (n-1) \cdot (n-2) \cdot \ldots \cdot (n-k+1) = \frac{n!}{(n-k)!}$$

A **permutation** of a n objects is an arrangement of each object (where order matters), so a **k-permutation** is an arrangement of k members of a set of n members (where order matters). The number of k-Permutations of n objects is just $P(n,k)$.

Example(s)

Suppose we have 13 chairs (in a row) with 9 TA's, and Professors Sunny, Rainy, Windy, and Cloudy to be seated. What is the number of seatings where every professor has a TA to his/her immediate left *and* right?

Solution This is quite a tricky problem if we don't choose the right setup. Imagine we first seat the 9 TA's - there are 9! ways to do this. Then, there are 8 spots between them, so that if we place a professor there, they're guaranteed to have a TA to their immediate left and right. We can't place more than one professor in a spot. Out of the 8 spots, we **pick** 4 of them for the professors to sit (order matters, since the professors are different people). So the answer by the product rule is $9! \cdot P(8,4)$. □

1.2.2 k-Combinations (Binomial Coefficients)

What if order *doesn't* matter? For example, if I need to **choose** 3 out of 7 weapons on my online adventure? We'll tackle that now, continuing our rainbow example!

A kindergartener smears 3 different colors out of 7 to make a new color. How many smeared colors can she create?

Notice that there are $3! = 6$ possible ways to order red, blue and orange, as you see below. However, all these rainbows produce the same "smeared" color!

Recall that there were $P(7,3) = 210$ possible mini-rainbows. But as we see from these rainbows, each "smeared" color is counted $3! = 6$ times. So, to get our answer, we take the 210 mini-rainbows and divide by 6 to account for the overcounting since in this case, order doesn't matter.

The answer is,

$$\frac{210}{6} = \frac{P(7,3)}{3!} = \frac{7!}{3!(7-3)!}$$

Definition 1.2.2: k-Combinations (Binomial Coefficients)

If we want to choose (order doesn't matter) **only** k out of n distinct objects, the number of ways to do so is $C(n,k) = \binom{n}{k}$ (read as "n choose k"), where

$$C(n,k) = \binom{n}{k} = \frac{P(n,k)}{k!} = \frac{n!}{k!(n-k)!}$$

A **k-combination** is a selection of k objects from a collection of n objects, in which the order does not matter. The number of k-Combinations of n objects is just $\binom{n}{k}$. $\binom{n}{k}$ is also called a **binomial coefficient** - we'll see why in the next section.

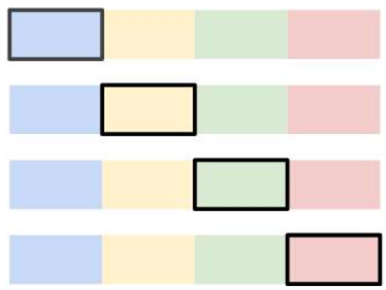
Notice, we can show from this that there is symmetry in the definition of binomial coefficients:

$$\binom{n}{k} = \frac{n!}{k!(n-k)!} = \frac{n!}{(n-k)!k!} = \binom{n}{n-k}$$

The algebra checks out - why is this true though, intuitively?

Let's suppose that $n = 4$ and $k = 1$. We want to show $\binom{4}{1} = \binom{4}{3}$. We have 4 colors:

These are the possible ways to choose 1 color out of 4:



These are the possible ways to choose 3 colors out of 4:

Looking at these, we can see that the color choices in each row are complementary. Intuitively, choosing 1 color is the same as choosing $4 - 1 = 3$ colors that we don't want - and vice versa. This explains the symmetry in binomial coefficients!

Example(s)

There are 6 AI professors and 7 theory professors taking part in an escape room. If 4 security professors and 4 theory professors are to be chosen and divided into 4 pairs (one AI professor with one theory professor per pair), how many pairings are possible?

Solution We first *choose* 4 out of 6 AI professors, with order not mattering, and 4 out of 7 theory professors, again with order not mattering. There are $\binom{6}{4} \cdot \binom{7}{4}$ ways to do this by the product rule. Then, for the first theory professor, we have 4 choices of AI professor to match with, for the second theory professor, we only have 3 choices, and so on. So we multiply by 4! to pair them off, and we get $\binom{6}{4} \cdot \binom{7}{4} \cdot 4!$. You may have counted it differently, but check if your answer matches! $\square$

1.2.3 Multinomial Coefficients

Now we'll see if we can generalize our binomial coefficients to solve even more interesting problems. Actually, they can be derived easily from binomial coefficients.

How many ways can you arrange the letters in "MATH"?

$4! = 24$, since they are distinct objects.

But if we want to rearrange the letters in "POOPOO", we have indistinct letters (two types - P and O). How do we approach this?

One approach is to choose where the 2 P's go, and then the O's have to go in the remaining 4 spots ($\binom{4}{4} = 1$ way) . Or, we can choose where the 4 O's go, and then the remaining P's are set ($\binom{2}{2} = 1$ way).

Either way, we get,

$$\binom{6}{2} \cdot \binom{4}{4} = \binom{6}{4} \cdot \binom{2}{2} = \frac{6!}{2!4!}$$

Another interpretation of this formula is that we are first arranging the 6 letters as if they were distinct: $P_1O_1O_2P_2O_3O_4$. Then, we divide by 4! and 2! to account for 4 duplicate O's and 2 duplicate P's.

What if we got even more complex, let's say three different letters? For example, rearranging the word 'BABYYYBAY'. There are 3 B's, 2 A's, and 4 Y's, for a total of 9 letters. We can choose where the 3 B's should go of the 9 spots: $\binom{9}{3}$ (order doesn't matter since all the B's are identical). Then out of the remaining 6 spots, we should choose 2 for the A's: $\binom{6}{2}$. Finally, out of the 4 remaining spots, we put the 4 Y's there: $\binom{4}{4} = 1$. By the product rule, our answer is

$$\binom{9}{3}\binom{6}{2}\binom{4}{4} = \frac{9!}{3!6!}\frac{6!}{2!4!}\frac{4!}{4!0!} = \frac{9!}{3!2!4!}$$

Note that we could have chosen to assign the Y's first instead: Out of 9 positions, we choose 4 to be Y: $\binom{9}{4}$. Then from the 5 remaining spots, choose where the 2 A's go: $\binom{5}{2}$, and the last three spots must be B's: $\binom{3}{3} = 1$. This gives us the equivalent answer

$$\binom{9}{4}\binom{5}{2}\binom{3}{3} = \frac{9!}{4!5!}\frac{5!}{2!3!}\frac{3!}{3!0!} = \frac{9!}{3!2!4!}$$

This shows once again that there are many correct ways to count something. This type of problem also frequently appears, and so we have a special notation (called a **multinomial coefficient**)

$$\binom{9}{3, 2, 4} = \frac{9!}{3!2!4!}$$

Note the order of the bottom three numbers does not matter (since the multiplication in the denominator is commutative), and that the bottom numbers must add up to the top number.

Definition 1.2.3: Multinomial Coefficients

If we have k types of objects (n total), with n_1 of the first type, n_2 of the second, ..., and n_k of the

k-th, then the number of arrangements possible is

$$\binom{n}{n_1, n_2, ..., n_k} = \frac{n!}{n_1!n_2!...n_k!}$$

This is a **multinomial coefficient**, the generalization of binomial coefficients.

Above, we had $k = 3$ objects (B, A, Y) with $n_1 = 3$ (number of B's), $n_2 = 2$ (number of A's), and $n_3 = 4$ (number of Y's), for an answer of $\binom{n}{n_1,n_2,n_3} = \frac{9!}{3!2!4!}$.

Example(s)

How many ways can you arrange the letters in "GODOGGY"?

Solution There are $n = 7$ letters. There are only $k = 4$ distinct letters - $\{G, O, D, Y\}$.
$n_1 = 3$ - there are 3 G's.
$n_2 = 2$ - there are 2 O's.
$n_3 = 1$ - there is 1 D.
$n_4 = 1$ - there is 1 Y.
This gives us the number of possible arrangements:

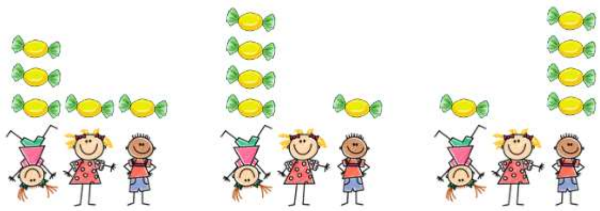
$$\binom{7}{3, 2, 1, 1} = \frac{7!}{3!2!1!1!}$$

It is important to note that even though the 1's are "useless" since $1! = 1$, we still must write every number on the bottom since they have to add to the top number. □

1.2.4 Stars and Bars/Divider Method

Now we tackle another common type of problem, which seems complicated at first. It turns out though that it can be reduced to binomial coefficients!

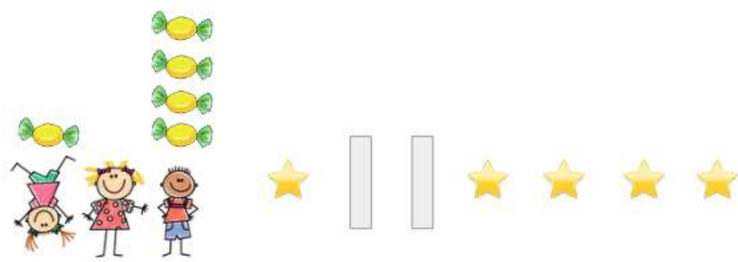
How many ways can we give 5 (indistinguishable) candies to these 3 (distinguishable) kids? Here are three possible distributions of candy:



Notice that the second and third pictures show different possible distributions, since the kids are distinguishable (different). Any idea on how we can tackle this problem?

The idea here is that we will count something equivalent. Let's say there are 5 "stars" for the 5 candies and 2 "bars" for the dividers (dividing 3 kids). For instance, this distribution of candies corresponds to this arrangement of 5 stars and 2 bars:

Here is another example of the correspondence between a distribution of candies and the arrangement of stars and bars:



For each candy distribution, there is exactly one corresponding way to arrange the stars and bars. Conversely, for each arrangement of stars and bars, there is exactly one candy distribution it represents.

Hence, the number of ways to distribute 5 candies to the 3 kids is the number of arrangements of 5 stars and 2 bars.

This is simply

$$\binom{7}{2} = \binom{7}{5} = \frac{7!}{2!5!}$$

Amazing right? We just reduced this candy distribution problem to reordering letters!

Definition 1.2.4: Stars and Bars/Divider Method

The number of ways to distribute n indistinguishable balls into k distinguishable bins is

$$\binom{n+(k-1)}{k-1} = \binom{n+(k-1)}{n}$$

since we set up n stars for the n balls, and $k-1$ bars dividing the k bins.

Example(s)

How many ways can we assign 20 students to 4 different professors? Assume the students are indistinguishable to the professors; who only care *how many* students they have, and not which ones.

Solution This is actually the perfect setup for stars and bars. We have 20 stars (students) and 3 bars (professors), and so our answer is $\binom{23}{3} = \binom{23}{20}$. □

1.2.5 Exercises

1. There are 40 seats and 40 students in a classroom. Suppose that the front row contains 10 seats, and there are 5 students who must sit in the front row in order to see the board clearly. How many seating arrangements are possible with this restriction?

   Solution: Again, there may be many correct approaches. We can first choose which 5 out of 10 seats in the front row we want to give, so we have $\binom{10}{5}$ ways of doing this. Then, assign those 5 students to these seats, to which there are 5! ways. Finally, assign the other 35 students in any way, for 35! ways. By the product rule, there are $\binom{10}{5} \cdot 5! \cdot 35!$ ways to do so.

2. If we roll a fair 3-sided die 11 times, what is the number of ways that we can get 4 1's, 5 2's, and 2 3's?

   Solution: We can write the outcomes as a sequence of length 11, each digit of which is 1, 2 or 3. Hence, the number of ways to get 4 1's, 5 2's, and 2 3's, is the number of orderings of 11112222233, which is $\binom{11}{4,5,2} = \frac{11!}{4!5!2!}$.

3. These two problems are almost identical, but have drastically different approaches to them. These are both extremely hard/tricky problems, though they may look deceivingly simple. These are probably the two coolest problems I've encountered in counting, as they do have elegant solutions!

   (a) How many 7-digit phone numbers are such that the numbers are strictly increasing (digits must go up)? (e.g., 014-5689, 134-6789, etc.)

   (b) How many 7-digit phone numbers are such that the numbers are monotone increasing (digits can stay the same or go up)? (e.g., 011-5566, 134-6789, etc.) Hint: Reduce this to stars and bars.

   Solution:

   (a) We choose 7 out of 10 digits, which has $\binom{10}{7}$ possibilities, and then once we do, there is only 1 valid ordering (must put them in increasing order). Hence, the answer is simply $\binom{10}{7}$. This question has a deceivingly simple solution, as many students (including myself at one point), would have started by choosing the first digit. But the choices for the next digit depend on the first digit. And so on for the third. This leads to a complicated, nearly unsolvable mess!

   (b) This is a very difficult problem to frame in terms of stars and bars. We need to map one phone number to exactly one ordering of stars and bars, and vice versa. Consider letting the 9 bars being an increase from one-digit to the next, and 7 stars for the 7 digits. This is extremely complicated, so we'll give 3 examples of what we mean.

      i. The phone number 011-5566 is represented as $*|**||||**|**|||$. We start a counter at 0, we see a digit first (a star), so we mark down 0. Then we see a bar, which tells us to increase our counter to 1. Then, two more digits (stars), which say to mark down 2 1's. Then, 4 bars which tell us to increase count from 1 to 5. Then two *'s for the next two 5's, and a bar to increase to 6. Then, two stars indicate to put down 2 6's. Then, we increment count to 9 but don't put down any more digits.

      ii. The phone number 134-6789 is represented as $|*||*|*||*|*|*|*$. We start a counter at 0, and we see a bar first, so we increase count to 1. Then a star tells us to actually write down 1 as our first digit. The two bars tell us to increase count from 1 to 3. The star says mark a 3 down now. Then, a bar to increase to 4. Then a star to write down 4. Two bars to increase to 6. And so on.

iii. The stars and bars ordering $|||| * | * * * *|| * ||*$ represents the phone number 455-5579. We start a counter at 0. We see 4 bars so we increment to 4. The star says to mark down a 4. Then increment count by 1 to 5 due to the next bar. Then, mark 5 down 4 times (4 stars). Then increment count by 2, put down a 7, and repeat to put down a 9.

Hence there is a bijection between these phone numbers and arrangements of 7 stars and 9 bars. So the number of satisfying phone numbers is $\binom{16}{7} = \binom{16}{9}$.

Chapter 1. Combinatorial Theory

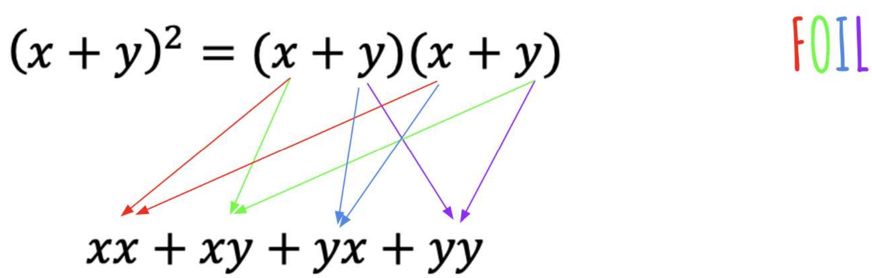
1.3: No More Counting Please

Slides (Google Drive) Video (YouTube)

In this section, we don't really have a nice successive ordering where one topic leads to the next as we did earlier. This section serves as a place to put all the final miscellaneous but useful concepts in counting.

1.3.1 Binomial Theorem

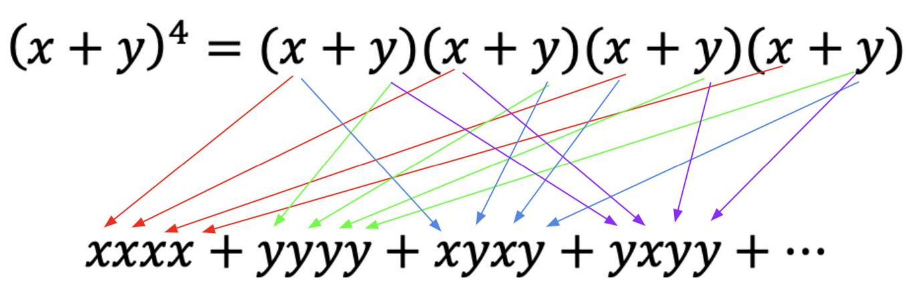
We talked last time about binomial coefficients of the form $\binom{n}{k}$. Today, we'll see how they are used to prove the binomial theorem, which we'll use more later on. For now, we'll see how they can be used to expand (possibly large) exponents below. You may have learned this technique of FOIL (first, outer, inner, last) for expanding $(x+y)^2$.



We then combine like-terms (xy and yx).

$$\begin{aligned}(x+y)^2 &= (x+y)(x+y) \\ &= xx + xy + yx + yy && \text{[FOIL]} \\ &= x^2 + 2xy + y^2\end{aligned}$$

But, let's say that we wanted to do this for a binomial raised to some higher power, say $(x+y)^4$. There would be a lot more terms, but we could use a similar approach.



$$\begin{aligned}(x+y)^4 &= (x+y)(x+y)(x+y)(x+y)\\ &= xxxx + yyyy + xyxy + yxyy + \dots\end{aligned}$$

But what are the terms exactly that are included in this expression? And how could we combine the like-terms though?

Notice that each term will be a mixture of x's and y's. In fact, each term will be in the form $x^k y^{n-k}$ (in this case $n = 4$). This is because there will be exactly n x's or y's in each term, so if there are k x's, then there must be $n - k$ y's. That is, we will have terms of the form $x^4, x^3y, x^2y^2, xy^3, y^4$, with most appearing more than once.

For a specific k though, how many times does $x^k y^{n-k}$ appear? For example, in the above case, take $k = 1$, then note that $xyyy = yxyy = yyxy = yyyx = xy^3$, so xy^3 will appear with the coefficient of 4 in the final simplified form (just like for $(x + y)^2$ the term xy appears with a coefficient 2). Does this look familiar? It should remind you yet again of rearranging words with duplicate letters!

Now, we can generalize this, as the number of terms will simplify to $x^k y^{n-k}$ will be equivalent to the number of ways to choose exactly k of the binomials to give us x (and let the remaining $n-k$ give us y). Alternatively, we need to arrange k x's and $n - k$ y's. To think of this in the above example with $k = 1$ and $n = 4$, we were consider which of the four binomials would give us the single x, the first, second, third, or fourth, for a total of $\binom{4}{1} = 4$.

Let's consider $k = 2$ in the above example. We want to know how many terms are equivalent to x^2y^2. Well, we then have $xxyy = yxxy = yyxx = xyxy = yxyx = xyyx = x^2y^2$, so there are six ways and the coefficient on the simplified term x^2y^2 will be $\binom{4}{2} = 6$.

Notice that we are essentially choosing which of the binomials gives us an x such that k of the n binomials do. That is, the coefficient for $x^k y^{n-k}$ where k ranges from 0 to n is simply $\binom{n}{k}$. This is why it was also called a binomial coefficient.

That leads us to the binomial theorem:

Theorem 1.3.1: Binomial Theorem

Let $x, y \in \mathbb{R}$ be real numbers and $n \in \mathbb{N}$ a positive integer. Then:

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}$$

This essentially states that in the expansion of the left side, the coefficient of the term with x raised to the power of k and y raised to the power of $n - k$ will be $\binom{n}{k}$, and we know this because we are considering the number of ways to choose k of the n binomials in the expression to give us x.

This can also be proved by induction, but this is left as an exercise for the reader.

Example(s)

Calculate the coefficient of $a^{45}b^{14}$ in the expansion $(4a^3 - 5b^2)^{22}$.

Solution Let $x = 4a^3$ and $y = -5b^2$. Then, we are looking for the coefficient of $x^{15}y^7$ (because x^{15} gives us a^{45} and y^7 gives us b^{14}), which is $\binom{22}{15}$. So we have the term

$$\binom{22}{15}x^{15}y^7 = \binom{22}{15}(4a^3)^{15}(-5b^2)^7 = \left(-\binom{22}{15}4^{15}5^7\right)a^{45}b^{14}$$

and our answer is $-\binom{22}{15}4^{15}5^7$. □

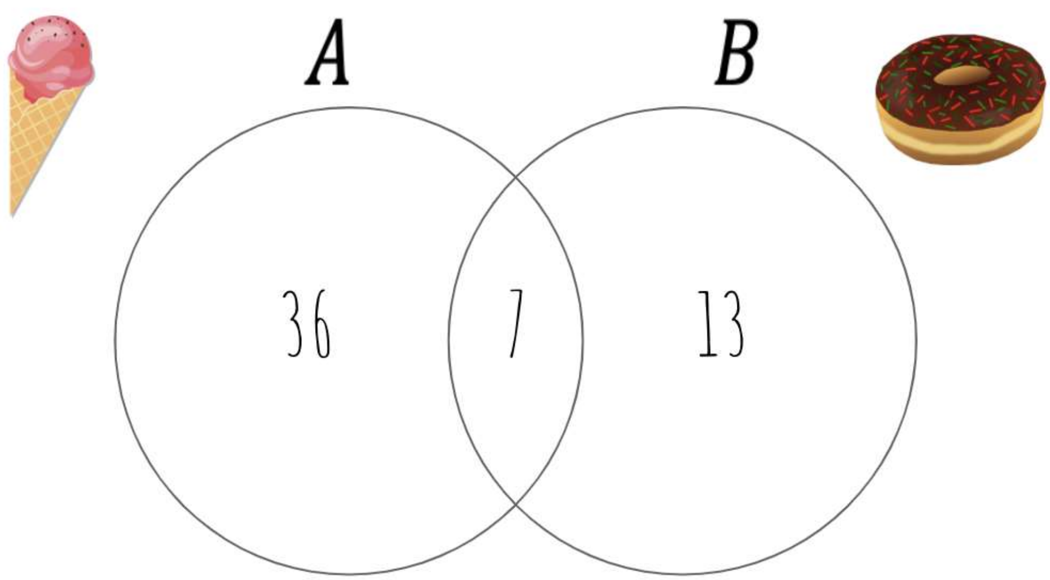
1.3.2 Inclusion-Exclusion

Say we did an anonymous survey where we asked whether students in CSE312 like ice cream, and found that 43 people liked ice cream. Then we did another anonymous survey where we asked whether students in CSE312 liked donuts, and found that 20 people liked donuts. With this information can we determine how many people like ice cream or donuts (or both)?

Let A be the set of people who like ice cream, and B the set of people who like donuts. The sum rule from 1.1 said that, if A, B were mutually exclusive (it wasn't possible to like both donuts and ice cream: $A \cap B = \emptyset$), then we could just add them up: $|A \cup B| = |A| + |B| = 43 + 20 = 63$. But this is not the case, since it is possible that to like both. We can't quite figure this out yet without knowing how many people overlapped: the size of $A \cap B$.

So, we did another anonymous survey in which we asked whether students in CSE312 like both ice cream and donuts, and found that only 7 people like both. Now, do we have enough information to determine how many students like ice cream or donuts?

Yes! Knowing that 43 people like ice cream and 7 people like both ice cream and donuts, we can conclude that 36 people like ice cream but don't like donuts. Similarly, knowing that 20 people like donuts and 7 people like both ice cream and donuts, we can conclude that 13 people like donuts but don't like ice cream. This leaves us with the following picture, where A is the students who like ice cream. B is the students who like donuts (this implies $|A \cap B| = 7$ is the number of students who like both):



So we have the following:

$$|A| = 43$$
$$|B| = 20$$
$$|A \cap B| = 7$$

Now, to go back to the question of how many students like either ice cream or donuts, we can just add up the 36 people that just like ice cream, the 7 people that like both ice cream and donuts, and the 13 people that just like donuts, and get $36 + 7 + 13 = 56$. Alternatively, we could consider this as adding up the 43 people who like ice cream (including both the 36 those who just like ice cream and the 7 who like both) and the 20 people who like donuts (including the 13 who just like donuts and the 7 who like both) and then subtracting the 7 who like both since they were counted twice. That is $43 + 20 - 7 = 56$. That leaves us with:

$$|A \cup B| = 36 + 7 + 13 = 56 = 43 - 20 - 7 = |A| + |B| - |A \cap B|$$

Recall that $|A \cup B|$ is the students who like donuts or ice cream (the union of the two sets).

Theorem 1.3.2: Inclusion-Exclusion

Let A, B be sets, then

$$|A \cup B| = |A| + |B| - |A \cap B|$$

Further, in general, if $A_1, A_2, \ldots, A_n$ are sets, then:

$$\begin{aligned} |A_1 \cup \cdots \cup A_n| &= \text{singles} - \text{doubles} + \text{triples} - \text{quads} + \cdots \\ &= (|A_1| + \cdots + |A_n|) - (|A_1 \cap A_2| + \cdots + |A_{n-1} \cap A_n|) \\ &\quad + (A_1 \cap A_2 \cap A_3| + \cdots + |A_{n-2} \cap A_{n-1} \cap A_n|) + \cdots \end{aligned}$$

where singles are the sizes of all the single sets ($\binom{n}{1}$ terms), doubles are the sizes of all the intersections of two sets ($\binom{n}{2}$ terms), triples are the size of all the intersections of three sets ($\binom{n}{3}$ terms), quads are all the intersection of four sets, and so forth.

Example(s)

How many numbers in the set $[360] = \{1, 2, \ldots, 360\}$ are divisible by:

1. 4, 6, and 9.
2. 4, 6 or 9.
3. neither 4, 6, nor 9.

Solution

1. This is just the multiplies of $\text{lcm}(4, 6, 9) = 36$, which there are $\dfrac{360}{36} = 10$ of.
2. Let D_i be the number of numbers in $[360]$ which are divisible by i, for $i = 4, 6, 9$. Hence, the number of numbers which are divisible by 4, 6, or 9 is $|D_4 \cup D_6 \cup D_9|$. We can apply inclusion-exclusion (singles

minus doubles plus triples):

$$\begin{aligned}|D_4 \cup D_6 \cup D_9| &= |D_4| + |D_6| + |D_9| - |D_4 \cap D_6| - |D_4 \cap D_9| - |D_6 \cap D_9| + |D_4 \cap D_6 \cap D_9| \\ &= \frac{360}{4} + \frac{360}{6} + \frac{360}{9} - \frac{360}{12} - \frac{360}{36} - \frac{360}{18} + \frac{360}{36}\end{aligned}$$

Notice the denominators for the paired terms are again, dividing by the least common multiple.

3. Complementary counting - this is just 360 minus the answer from the previous part!

□

Many times it may be possible to avoid this ugly mess using complementary counting, but sometimes it isn't.

1.3.3 Pigeonhole Principle

The Pigeonhole Principle is a tool that allows us to make guarantees when we tackle problems like: if we want to assign 20 third-grade students to 3 (equivalent) classes, how can we minimize the largest class size? It turns out we can't do any better than having 7 people in the largest class. The reason is because of the pigeonhole principle!

We'll start with a smaller but similar problem. If 11 children have to share 3 beds, how can we minimize the number of children on the most crowded bed? The idea might be just to spread them "uniformly". Maybe number the beds A,B,C, and assign the first child to A, the second to B, the third to C, the fourth to A, and so on. This turns out to be optimal as it spreads the kids out as evenly as possible. The pigeonhole principle tells us the best worst-case scenario: that at least one bed must have at least 4 children.

You might first distribute the children evenly amongst the beds, say put 3 children in each bed to start. That leaves us with 3 times 3 equals 9 children accounted for, and 2 children remaining with a bed. Well, they must be put to bed, so we can put each of them in a separate bed and we finish with the first bed having 4, the second bed having 4, and the third bed having 3. No matter how we move the children around, we can't have an an arrangement where at least one bed will contain at least 4 children.

We could also have found this by dividing 11 by 3 and rounding up to account for the remainder (which must go somewhere). Before formally defining the pigeonhole principle, we need to define the floor and ceiling functions.

Definition 1.3.1: Floor and Ceiling Functions

The **floor** function $\lfloor x \rfloor$ returns the largest integer $\leq x$ (i.e. rounds down).
The **ceiling** function $\lceil x \rceil$ returns the smallest integer $\geq x$ (i.e. rounds up). Note the difference is just whether the bracket is on top (ceiling) or bottom (floor).

Example(s)

Solve the following: $\lfloor 2.5 \rfloor$, $\lfloor 16.999999 \rfloor$, $\lfloor 5 \rfloor$, $\lceil 2.5 \rceil$, $\lceil 9.000301 \rceil$, $\lceil 5 \rceil$.

Solution

$$\lfloor 2.5 \rfloor = 2 \qquad \lfloor 16.999999 \rfloor = 16 \qquad \lfloor 5 \rfloor = 5$$
$$\lceil 2.5 \rceil = 3 \qquad \lceil 9.000301 \rceil = 10 \qquad \lceil 5 \rceil = 5$$

□

Theorem 1.3.3: Pigeonhole Principle

If there are n pigeons we want to put into k holes (where $n > k$), then at least one pigeonhole must contain at least 2 pigeons.
More generally, if there are n pigeons we want to put into k pigeonholes, then at least one pigeonhole must contain at least $\lceil n/k \rceil$ pigeons.

This fact or rule may seem trivial to you, but the hard part of pigeonhole problems is knowing how to apply it. See the examples below!

Example(s)

Show that there exists a number made up of only 1's (e.g., 1111 or 11) which is divisible by 333.

Solution Consider the sequence of 334 numbers $x_1, x_2, x_3, \ldots, x_{334}$ where x_i is the number made of exactly i 1's (e.g., $x_2 = 11$, $x_5 = 11,111$, etc.). We'll use the notation $x_i = 1^i$ to mean i 1's concatenated together.

The number of possible remainders when dividing by 333 is 333: $\{0, 1, 2, \ldots, 332\}$, so by the pigeonhole principle, since $334 > 333$, two numbers x_i and x_j have the same remainder (suppose $i < j$ without loss of generality) when divided by 333. The number $x_j - x_i$ is of the form $1^{j-i}0^i$; that is $j - i$ 1's followed by i 0's (e.g., $x_5 - x_2 = 11111 - 11 = 11100 = 1^3 0^2$). This number must be divisible by 333 because $x_i \equiv x_j \pmod{333} \Rightarrow (x_j - x_i) \equiv 0 \pmod{333}$.

Now, keep deleting zeros (by dividing by 10) until there aren't any more left - this doesn't affect whether or not 333 goes in since neither 2 nor 5 divides 333. Now we're left with a number divisible by 333 made up of all ones (1^{j-i} to be exact)!

Note that 333 was not special - we could have used any number that wasn't divisible by 2 nor 5. □

Example(s)

Show that in a group of n people (who may be friends with any number of other people), two must have the same number of friends.

Solution We have two cases.

1. Case 1: Everyone has at least one friend. Then, everyone has a number of friends between $1, 2, \ldots, n-1$. By the pigeonhole principle, since there are n people and $n - 1$ possibilities, at least two people have the same number of friends.
2. Case 2: At least one person has no friends. Let's take one such person and call them A. Then, the other $n-1$ people can have number of friends from $0, \ldots, n-2$ since they can't be friends with A. We have two more cases within this case unfortunately.
   (a) Case 2a: If one of these $n - 1$ people has no friends, we are done since A and this person both have 0 friends.
   (b) Case 2b: Otherwise, these people all have at least one friend, from $1, \ldots, n - 2$, and since there are $n - 1$ people and $n - 2$ possibilities, at least two people have the same number of friends.

   In all cases we are guaranteed that two people have the same number of friends.

□

1.3.4 Combinatorial Proofs

You may have taken a discrete mathematics/formal logic class before this if you are a computer science major. If that's the case, you would have learned how to write proofs (e.g., induction, contradiction). Now that we know how to count, we can actually prove some algebraic identities using counting instead!

Suppose we wanted to show that $\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$ was true for any positive integer $n \in \mathbb{N}$ and $0 \le k \le n$.

We could start with an algebraic approach and try something like:

$$\begin{aligned}
\binom{n-1}{k-1} + \binom{n-1}{k} &= \frac{(n-1)}{(k-1)!(n-k)!} + \frac{(n-1)}{k!(n-1-k)!} && \text{[def of binomial coef]} \\
&\cdots && \text{[lots of algebra]} \\
&= \frac{n!}{k!(n-k)!} \\
&= \binom{n}{k}
\end{aligned}$$

However, those $\cdots$ may be tedious and take a lot of algebra we don't want to do.

So, let's consider another approach. A combinatorial proof is one where you prove two quantities are equal by imagining a situation/something to count. Then, you argue that the left side and right side are two equivalent ways to count the same thing, and hence must be equal. We've seen earlier often how there are multiple approaches to counting!

In this case, let's consider the set of numbers $[n] = \{1, 2, \ldots n\}$. We will argue that the LHS and RHS both count the number of subsets of size k.

1. LHS: $\binom{n}{k}$ is literally the number of subsets of size k, since we just want to choose any k items out of n (order doesn't matter).

2. RHS: We take a slightly more convoluted approach, splitting on cases depending on whether or not the number 1 was included in the subset.

   Case 1: Our subset of size k includes the number 1. Then we need to choose $k-1$ of the remaining $n-1$ numbers (n numbers excluding 1 is $n-1$ numbers) to make a subset of size k which includes 1.

   Case 2: Our subset of size k does not include the number 1. Then we need to choose k numbers from the remaining $n-1$ numbers. There are $\binom{n-1}{k}$ ways to do this. So, in total we have $\binom{n-1}{k-1} + \binom{n-1}{k}$ possible subsets of size k.

Since the left side and right side count the same thing, they must be equal! Note that we dreamed up this situation and you may wonder how we did - this just comes from practicing many types of counting problems. You'll get used to it!

Definition 1.3.2: Combinatorial Proofs

To prove two quantities are equal, you can come up with a combinatorial situation, and show that both in fact count the same thing, and hence must be equal.

Example(s)

Prove the following two identities combinatorially (NOT algebraically):

1. Prove that $\binom{n}{m}\binom{m}{k} = \binom{n}{k}\binom{n-k}{m-k}$.
2. Prove that $2^n = \sum_{k=0}^{n} \binom{n}{k}$

Solution

1. We'll show that both sides count, from a group of n people, the number of committees of size m, and within that committee a subcommittee of size k.

   Left-hand side: We first choose m people to be on the committee from n total; there are $\binom{n}{m}$ ways to do so. Then, within those m, we choose k to be on a specialized subcommittee; there are $\binom{m}{k}$ ways to do so. By the product rule, the number of ways to assign these is $\binom{n}{m}\binom{m}{k}$.

   Right-hand side: We first choose which k to be on the subcommitee of size k; there are $\binom{n}{k}$ ways to do so. From the remaining $n - k$ people, we choose $m - k$ to be on the committee (but not the subcommittee). By the product rule, the number of ways to assign these is $\binom{n}{k}\binom{n-k}{m-k}$.

   Since the LHS and RHS both count the same thing, they must be equal.

2. We'll argue that both sides count the number of subsets of the set $[n] = \{1, 2, \ldots, n\}$.

   Left-hand side: Each element we can have in our subset or not. For the first element, we have 2 choices (in or out). For the second element, we also have 2 choices (in or out). And so on. So the number of subsets is 2^n.

   Right-hand side: The subset can be of any size ranging from 0 to n, so we have a sum. Now how many subsets are there of size exactly k? There are $\binom{n}{k}$ because we choose k out of n to have in our set (and order doesn't matter in sets)! Hence, the number of subsets is $\sum_{k=0}^{n} \binom{n}{k}$

   Since the LHS and RHS both count the same thing, they must be equal.

   It's cool to note we can also prove this with the binomial theorem setting $x = 1$ and $y = 1$ - try this out! It takes just one line!

□

1.3.5 Exercises

1. These problems involve using the pigeonhole principle. How many cards must you draw from a standard 52-card deck (4 suits and 13 cards of each suit) until you are guaranteed to have:
   (a) A single pair? (e.g., AA, 99, JJ)
   (b) Two (different) pairs? (e.g., AAKK, 9933, 44QQ)
   (c) A full house (a triple and a pair)? (e.g., AAAKK, 99922, 555JJ)
   (d) A straight (5 in a row, with the lowest being A,2,3,4,5 and the highest being 10,J,Q,K,A)?
   (e) A flush (5 cards of the same suit)? (e.g., 5 hearts, 5 diamonds)
   (f) A straight flush (5 cards which are both a straight and a flush)?

Solution:

(a) The worst that could happen is to draw 13 different cards, but the next is guaranteed to form a pair. So the answer is 14.

(b) The worst that could happen is to draw 13 different cards, but the next is guaranteed to form a pair. But then we could draw the other two of that pair as well to get 16 still without two pairs. So the answer is 17.

(c) The worst that could happen is to draw all pairs (26 cards). Then the next is guaranteed to cause a triple. So the answer is 27.

(d) The worst that could happen is to draw all the A - 4, 6 - 9, and J - K. After drawing these $11 \cdot 4 = 44$ cards, we could still fail to have a straight. Finally, getting a 5 or 10 would give us a straight. So the answer is 45.

(e) The worst that could happen is to draw 4 of each suit (16 cards), and still not have a flush. So the answer is 17.

(f) Same as straight, 45.